J. J. Blanco-Pillado*, Ken D. Olum† and A. Vilenkin‡
Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, Massachusetts 02155
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We give a general solution to the equations of motion for superconducting strings with chiral (null) currents not coupled to any long-range fields. We apply this solution to show that the motion of such string loops is strictly periodic, and we briefly analyze cusp-like behavior and vorton-like solutions of arbitrary shape.

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Introduction Cosmic strings are linear topological defects that may have been created in the early Universe [1]. They have been extensively studied in connection with several problems in cosmology (for reviews see [2,3]). Assuming that the radius of curvature of a string is always much larger than the string core thickness, the string dynamics can be described by the Nambu-Goto action [4,5]. In this approximation the equations of motion are simple, and can be easily solved.

In 1985, Witten [6] showed that strings could behave as superconducting wires in certain particle physics models. This new internal degree of freedom opened up a variety of interesting effects. In particular, superconducting strings may have stable configurations, vortons [7] and springs [8,9], which could contribute to the dark matter in the universe, or put constraints on the particle physics models that give rise to those strings [10]. Superconducting cosmic strings have also been considered as sources for structure formation [11], gamma ray bursts [12–14], and ultra-high energy cosmic rays [14–16].

The general problem of a superconducting string coupled to the electromagnetic field cannot be solved analytically. However, if the charge carriers are not coupled to any long-range field (so-called neutral superconducting strings) [17] the situation is significantly simpler. Such strings are of interest in their own right, and may also be considered as approximations to the full theory including electromagnetic coupling.

If in addition, we consider the case that the charge and current are equal in magnitude, then the equations of motion can be solved exactly, as we show below. In this case, the charge-current 4-vector is lightlike, and the current is said to be chiral or null. It consists of charge carriers which are all moving in the same direction. Such currents arise automatically in certain supersymmetric

*Email address: `jose@cosmos2.phy.tufts.edu`
†Email address: `kdo@alum.mit.edu`
‡Email address:`vilenkin@cosmos2.phy.tufts.edu`
*Email address: `jose@cosmos2.phy.tufts.edu`
†Email address: `kdo@alum.mit.edu`
‡Email address: `vilenkin@cosmos2.phy.tufts.edu`

theories in which a zero mode can travel only in one direction along the string [19,10]. They could also result from evolution of a loop with an arbitrary distribution of charge and current [7,20], because ejection of charge carriers will drive the string toward the chiral limit. This will happen particularly near a cusp, since the large Lorentz factor would produce the ejection of charge carriers and bring the string to the chiral limit very efficiently.

After this work was completed, we learned of work by Carter and Peter [18] which found a very similar "integrability property" for chiral strings. However, we derive our results here using an elementary formalism, and we go on to give an explicit solution for the initial value problem. We also discuss several physical implications of the result.

Nambu-Goto strings We first review the equations of motion and their solution in the case of non-superconducting strings. We will use similar techniques in the next section to solve the equations of motion for superconducting strings, in the case that the current is chiral.

For an infinitely thin non-superconducting relativistic string, the flat spacetime equations of motion are [2]

$$\partial_a \left[\sqrt{-\gamma}\, \gamma^{ab}\, x^{\nu}_{,b}\right] = 0 \tag{1}$$

where a and b take the values 0 and 1 (denoting the worldsheet coordinates τ and σ respectively), $x^{\nu}(\sigma,\tau)$ is the position of the string, and γ_{ab} is the induced metric on the worldsheet,

$$\gamma_{ab} = x^{\mu}_{,a} x_{\mu,b} \tag{2}$$

and $\gamma = \det(\gamma_{ab})$. We are free to choose a particular parameterization of the worldsheet, i.e. gauge condition. In this case, it is convenient to use the conformal gauge, namely

$$\gamma_{ab} = \Omega(\sigma,\tau)\, \eta_{ab} \tag{3}$$

where η_{ab} is the two-dimensional Minkowski metric. Then $\sqrt{-\gamma} = \Omega$ and thus $\sqrt{-\gamma}\gamma^{ab} = \eta^{ab}$. In this gauge the equation of motion for the string is just the two dimensional wave equation,

$$x''^{\nu} - \ddot{x}^{\nu} = 0\,, \tag{4}$$

where $\dot{x}$ denotes $\partial x/\partial\tau$ and x' denotes $\partial x/\partial\sigma$.

It can be seen that the constraints written above in Eq. (3), do not fix completely our gauge, and we can choose the condition

$$x^0 = \tau \tag{5}$$

so the equations of motion become

$$\mathbf{x}'' - \ddot{\mathbf{x}} = 0\,. \tag{6}$$

The general solution has the form

$$\mathbf{x} = \frac{1}{2}[\mathbf{a}(\sigma - \tau) + \mathbf{b}(\sigma + \tau)]\,. \tag{7}$$

In order for the metric to have the form of Eq. (3), we must impose the conditions

$$|\mathbf{a}'|^2 = |\mathbf{b}'|^2 = 1\,. \tag{8}$$

Chiral superconducting strings The equations of motion for a superconducting string with a neutral current (i.e., one not coupled to the electromagnetic field) can be written [6,21]

$$\partial_a \left(\mathcal{T}^{ab} x^\nu_{,b}\right) = 0 \tag{9a}$$

$$\partial_a \left(\sqrt{-\gamma}\gamma^{ab}\phi_{,b}\right) = 0 \tag{9b}$$

where

$$\mathcal{T}^{ab} = \sqrt{-\gamma}\,\left(\mu\gamma^{ab} + \theta^{ab}\right), \tag{10}$$

μ is the energy per unit length of the string, θ^{ab} the energy-momentum tensor of the charge carriers,

$$\theta^{ab} = \gamma^{ac}\gamma^{bd}\phi_{,c}\,\phi_{,d} - \frac{1}{2}\,\gamma^{ab}\gamma^{cd}\,\phi_{,c}\,\phi_{,d}\,, \tag{11}$$

and ϕ the auxiliary scalar field in terms of which we can express the conserved worldsheet current

$$J^a = \frac{1}{\sqrt{-\gamma}}\,\epsilon^{ab}\,\phi_{,b}\,. \tag{12}$$

The current is chiral if $\phi_{,a}$ is a null worldsheet vector, i.e.,

$$\gamma^{ab}\phi_{,b}\phi_{,a} = 0\,, \tag{13}$$

in which case

$$\theta^{ab} = \gamma^{ac}\gamma^{bd}\phi_{,c}\,\phi_{,d}\,. \tag{14}$$

As above, we would like to have a gauge in which the $\mathcal{T}^{ab}$ has the form

$$\mathcal{T}^{ab} = \mu\,\eta^{ab}\,. \tag{15}$$

If we can accomplish this, the equation of motion will be the wave equation, Eq. (4), we can choose $x^0 = \tau$, and the general solution will be given by Eq. (7), as before. However, we note that since $\mathcal{T}$ is a 2×2 matrix,

$$\begin{aligned}\det\mathcal{T} &= (-\gamma)\det\left(\mu\gamma^{ab} + \theta^{ab}\right)\\ &= -\det\left[\gamma_{ab}\left(\mu\gamma^{bc} + \theta^{bc}\right)\right] = -\det\left(\mu\delta^c_a + \theta^c_a\right)\end{aligned} \tag{16}$$

which is gauge-invariant. Since the matrices are 2×2, the determinant is easily expanded,

$$\det\mathcal{T} = -\mu^2 - \mu\,\mathrm{Tr}\,\theta^c_a - \det\theta^c_a\,. \tag{17}$$

Since θ is traceless, $\det\mathcal{T}$ can only be $-\mu^2$ as required by Eq. (15), if $\det\theta^c_a = 0$. For a general current, from Eq. (11),

$$\theta^c_a = \phi^{,c}\phi_{,a} - \frac{1}{2}\delta^c_a \phi^{,b}\phi_{,b}\,. \tag{18}$$

The first term is the outer product of two vectors, so its determinant vanishes. Then, using the same expansion as in Eq. (17), we can compute

$$\det \theta^c_a = \frac{1}{4}\left(\phi^{,b}\phi_{,b}\right)^2 - \frac{1}{2}\phi^{,b}\phi_{,b}\,\mathrm{Tr}\left(\phi^{,c}\phi_{,a}\right) = -\frac{1}{4}\left(\phi^{,b}\phi_{,b}\right)^2 \tag{19}$$

which vanishes if and only if the current is chiral.

Thus for a chiral current there is the possibility that Eq. (15) can be satisfied. In that case, from Eqs. (15) and (10) we see that

$$\sqrt{-\gamma}\gamma^{ab}\phi_{,b} = \frac{1}{\mu}\left[\mathcal{T}^{ab} - \theta^{ab}\right]\phi_{,b} = \eta^{ab}\phi_{,b} \tag{20}$$

and so Eq. (9b) becomes the wave equation,

$$\ddot{\phi} - \phi'' = 0\,. \tag{21}$$

We now contract Eq. (15) with $\phi_{,a}\phi_{,b}$. Since the current is chiral, $\gamma^{ab}\phi_{,a}\phi_{,b} = 0$, and using Eq. (14), $\theta^{ab}\phi_{,a}\phi_{,b} = 0$. Thus to satisfy Eq. (15), we must have $\eta^{ab}\phi_{,a}\phi_{,b} = 0$, or $\dot{\phi}^2 = \phi'^2$. Without loss of generality we take the solution of the form

$$\phi(\sigma,\tau) = F(\sigma+\tau)\,, \tag{22}$$

which also satisfies Eq. (21).

Using Eq. (22), the condition for chirality, Eq. (13), becomes

$$\gamma^{00} + \gamma^{11} + 2\gamma^{01} = 0\,. \tag{23}$$

Since γ^{ab} is the inverse of the 2×2 matrix γ_{ab} we have

$$\gamma^{ab} = \frac{1}{\gamma}\begin{pmatrix} \gamma_{11} & -\gamma_{01} \\ -\gamma_{01} & \gamma_{00} \end{pmatrix}, \tag{24}$$

so Eq. (23) implies

$$\gamma_{00} + \gamma_{11} - 2\gamma_{01} = 0\,. \tag{25}$$

Using Eq. (2), this becomes

$$0 = \dot{x}^\mu\dot{x}_\mu + x'^\mu x'_\mu - 2\dot{x}^\mu x'_\mu = (\dot{x}^\mu - x'^\mu)(\dot{x}_\mu - x'_\mu) \tag{26}$$

which means that $\dot{x}^\mu - x'^\mu = (1, -\mathbf{a}')$ is a null 4-vector, or that

$$|\mathbf{a}'| = 1\,. \tag{27}$$

To solve the rest of the problem, we define a matrix

$$\mathcal{S}_{ab} = \frac{1}{\sqrt{-\gamma}}\left(\mu\gamma_{ab} - \theta_{ab}\right)\,. \tag{28}$$

From Eqs. (13) and (14), $\theta_{ab}\theta^{bc} = 0$, and since γ^{ab} and θ^{ab} are symmetrical,

$$\mathcal{S}_{ab}\mathcal{T}^{bc} = \mu^2\,\delta_a^c \tag{29}$$

and consequently if $\mathcal{T}$ has the form of Eq. (15) we have that

$$\mathcal{S}_{ab} = \mu\eta_{ab}\,. \tag{30}$$

Now, comparing the 01 component of Eqs. (28) and (30) gives

$$\mu\gamma_{01} = \theta_{01} = F'^2\,. \tag{31}$$

The metric component is

$$\gamma_{01} = \dot{x}^\mu x'_\mu = \frac{1}{4}\left(|\mathbf{a}'|^2 - |\mathbf{b}'|^2\right)\,. \tag{32}$$

From Eqs. (32) and (27), we find

$$1 - |\mathbf{b}'|^2 = \frac{4\,F'^2}{\mu} \tag{33}$$

Since the determinant of $\mathcal{T}$ and thus of $\mathcal{S}$ is fixed, it remains only to show that one more component of Eq. (30) is satisfied. For example, a sufficient condition is that

$$\mu\gamma_{00} - \theta_{00} = \mu\sqrt{-\gamma}\,. \tag{34}$$

Using Eq. (25) it is easy to show that

$$\sqrt{-\gamma} = \frac{1}{2}\left(\gamma_{00} - \gamma_{11}\right), \tag{35}$$

and Eq. (34) becomes

$$\mu\gamma_{00} - F'^2 = \frac{\mu}{2}(\gamma_{00} - \gamma_{11}) \tag{36}$$

so

$$\frac{\mu}{2}(\gamma_{00} + \gamma_{11}) = F'^2 \tag{37}$$

which is satisfied using Eq. (25) and Eq. (31).

Thus, Eq. (7) with the constraints given by Eqs. (27) and (33) are a general solution to the equations of motion for a superconducting string with a chiral current.

Note that in this gauge σ parameterizes the total energy on the string. The energy in a region is

$$E = \int d^3x\,T_0^0\,, \tag{38}$$

where T^μ_ν is given by

$$T^\mu_\nu(x) = \int d\sigma\,d\tau\,\mathcal{T}^{ab}x^\mu_{,a}\,x_{\nu,b}\,\delta^4[x - x(\sigma,\tau)]. \tag{39}$$

With $x^0 = \tau$, and using Eq. (15), the energy is

$$E = \mu\int d\sigma\; = \mu\int dl\frac{1}{|\mathbf{x}'|} \tag{40}$$

which means that with this parameterization the energy on the string is equal $\mu\,\Delta\sigma$.

Finding the solution from initial conditions. We would now like to find the evolution of a string with a chiral current from given initial conditions. We suppose that we are given the position of the string at some time t_0, as a function $\mathbf{x}(l)$ parameterized by arc length in the laboratory frame, with l increasing in the direction of current flow. We also need the initial charge and current as the values of the auxiliary scalar field $\phi(l)$, and the perpendicular component of the string motion, $\dot{\mathbf{x}}_\perp(l)$. Motion parallel to the string direction is dependent on the choice of parameter and has no physical meaning. From these conditions, we want to find the functions $\mathbf{a}$ and $\mathbf{b}$.

The first step is to reparameterize everything in terms of σ. For a stationary string, the linear energy density of the string itself is just μ, and the energy due to the current is $(d\phi/dl)^2$. Boosting the string in a transverse direction just gives the Lorentz factor $\Gamma = 1/\sqrt{1-|\dot{\mathbf{x}}_\perp|^2}$, so

$$\frac{dE}{dl} = \Gamma\left[\mu + \left(\frac{d\phi}{dl}\right)^2\right] \tag{41}$$

and so

$$\frac{d\sigma}{dl} = \Gamma\left[1 + \frac{1}{\mu}\left(\frac{d\phi}{dl}\right)^2\right]. \tag{42}$$

Using Eq. (42) we can change parameters from l to σ.

Now we need to determine the full form of $\dot{\mathbf{x}}$. We observe that $\dot{\mathbf{x}}\cdot\mathbf{x}' = (|\mathbf{b}'|^2 - |\mathbf{a}'|^2)/4 = -\phi'^2/\mu$, so we can write

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_\perp - \frac{\phi'^2\mathbf{x}'}{\mu\,|\mathbf{x}'|^2}. \tag{43}$$

Then

$$\mathbf{a}' = \mathbf{x}' - \dot{\mathbf{x}} \tag{44a}$$
$$\mathbf{b}' = \mathbf{x}' + \dot{\mathbf{x}} \tag{44b}$$

and Eq. (7) gives a complete solution for the future evolution of the string.

Discussion We have obtained the analytic general solution for superconducting strings with chiral currents. For a string with a current given by

$$\phi(\sigma,\tau) = F(\sigma+\tau)\,, \tag{45}$$

the general solution for the string is given by

$$x^0 = \tau \tag{46a}$$
$$\mathbf{x} = \frac{1}{2}[\mathbf{a}(\sigma-\tau) + \mathbf{b}(\sigma+\tau)]\,, \tag{46b}$$

with the following constraints for the otherwise arbitrary functions $\mathbf{a}'$ and $\mathbf{b}'$,

$$|\mathbf{a}'|^2 = 1 \tag{47a}$$
$$|\mathbf{b}'|^2 = 1 - \frac{4\,F'^2}{\mu}. \tag{47b}$$

We now show several interesting consequences that we can extract from this result. First of all, we see that there are arbitrarily shaped static solutions for the case in which the current satisfies $4\,{F'}^2/\mu = 1$. In this case $|\mathbf{b}'| = 0$, so the position of the string, up to a constant vector, is given by

$$\mathbf{x} = \frac{1}{2}[\mathbf{a}(\sigma - \tau)] \tag{48}$$

so the set of points traced by the string does not depend on time. These are vortons of any shape, which could have important cosmological consequences.

We also see that chiral strings do not have true cusps. In fact, we can easily calculate the maximum Lorentz factor that these strings can reach,

$$\Gamma = \frac{1}{\sqrt{1-\dot{\mathbf{x}}_\perp^2}} = \sqrt{1 + \left(\frac{|\mathbf{b}'|\sin\theta}{1+|\mathbf{b}'|\cos\theta}\right)^2}\,, \tag{49}$$

where θ is the angle between $\mathbf{a}'$ and $\mathbf{b}'$. This expression has its maximum at $\cos\theta = -|\mathbf{b}'|$, and the maximum Lorentz factor is

$$\Gamma_{\max} = \frac{\sqrt{\mu}}{2|F'|}\,. \tag{50}$$

On the other hand, the maximum concentration of energy per unit length (including contributions from both the string and the charge carriers) does not occur at the same point in the string evolution. Using Eq. (40) we see that the maximum energy density corresponds to the minimum of $|\mathbf{x}'|$, i.e. $\theta = \pi$. There,

$$|\mathbf{x}'|_{\min} = \frac{1}{2}\left(1 - |\mathbf{b}'|\right) \tag{51}$$

and so

$$\left.\frac{dE}{dl}\right|_{\max} = \frac{2\mu}{1-|\mathbf{b}'|} = \frac{2\mu}{1-\sqrt{1-4F'^2/\mu}}\,. \tag{52}$$

Another somewhat unexpected consequence of the exact solution is that the motion of a loop with a chiral current is strictly periodic in its rest frame. The period is $T = E/(2\mu)$, where E is the total energy of the loop in that frame.

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